On average non-terminal occurrences in regular tree languages

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1 The Problem

Comon et al. [Com+07] define the class of *regular tree languages* tREG to be the sets of trees generated by *regular tree grammars*[1]. Such a grammar $G = (N, \Sigma, \delta, S)$ is similar to the grammars we know from word languages, i. e. we have finite sets N and Σ of non-terminals and terminals, respectively, a finite rule set δ and a start symbol $S \in N$. W. l. o. g. we can assume G to be *reduced* and *normalized* [Com+07, Propositions 2.1.3 and 2.1.4] as well as *unambiguous* (cf. Lemma 3). That is we can assume that all rules are of the form

$$A \to \begin{array}{c} \mathrm{a} \\ / \quad \backslash \\ A_1 \cdots A_k \end{array}$$

or

$$A \to a$$

with $A, A_1, \dots, A_k \in N$ and $a \in \Sigma$, all non-terminals can be reached from S and every non-terminal can be replaced with a terminal (sub)tree by subsequent rule application. The notions of derivation and the generated language $L(G)$ carry over naturally from word grammars [Com+07, Section 2.1].

Given $L \in \mathsf{tREG}$, define

$$L_n := \{T \in L \mid |T| = n\},$$

where $|T|$ denotes the number of nodes in T, and

$$\overline{H}_n := \frac{1}{|L_n|} \cdot \sum_{T \in L_n} \operatorname{height}(T) \, .$$

[1] Actually, they use tree automata first, but let's stick with grammars for our purposes.

The question is now:

> Are there regular tree languages L with average height
>
> $$\overline{H}_n \notin \Theta(\sqrt{n}) \cup \Theta(n) \ ?$$

We know examples with $\overline{H}_n \in \Theta(\sqrt{n})$, e. g. all *simple varieties* [FS09, section VII.3], and $\overline{H}_n \in \Theta(n)$, e. g. unary chains, but no others.

Since height is not an additive parameter and hence hard to analyse, we have been trying to find ways to partition the problem. One idea has been to investigate the average height contribution

$$\overline{H}_{A,n} := \frac{1}{|L_n|} \cdot \sum_{T \in L_n} \operatorname{height}_A(T)$$

for each non-terminal A, where $\operatorname{height}_A(T)$ is the maximum number of terminals on a path from the root to a leaf in T that was generated by a rule with left-hand side A.

We note that the set of trees that remain after removing all non-A nodes from L (we obtain graph-theoretic *minors*) is always a simple variety; that is, these minors have either linear or square-root average height. However, L does usually not induce a uniform distribution on these minors, so the result does not carry over immediately. As a way to investigate how the contributions can mix, we have therefore been asking:

> Which forms can the average number of A-nodes, $A \in N$, defined as
>
> $$\overline{A}_n := \frac{1}{|L_n|} \cdot \sum_{T \in L_n} |T|_A \ ,$$
>
> have?

We now present two of the examples we have investigated.

2 Example Mbm

The first example has been proposed by Mireille Bousquet-Mélou. Trees consist of full binary trees on the left and unary chains on the right. We control the number of node labels (i. e. available terminal symbols); there are $c \in \mathbb{N}_+$ distinct labels for the full binary trees in the left, and $d \in \mathbb{N}_+$ distinct labels for the unary chains in the right subtree. This corresponds to the grammar induced by these rules:

$$S \to \begin{array}{c} a \\ / \ \backslash \\ \mathrm{A}\ \ \mathrm{B} \end{array}$$

$$A \to \begin{array}{c} a_1 \\ / \ \backslash \\ \mathrm{A}\ \ \mathrm{A} \end{array} \Bigg| \cdots \Bigg| \begin{array}{c} a_c \\ / \ \backslash \\ \mathrm{A}\ \ \mathrm{A} \end{array} \Bigg| a_1 \Bigg| \cdots \Bigg| a_c$$

$$B \to \begin{array}{c} a_1 \\ | \\ \mathrm{B} \end{array} \Bigg| \cdots \Bigg| \begin{array}{c} a_d \\ | \\ \mathrm{B} \end{array} \Bigg| a_1 \Bigg| \cdots \Bigg| a_d$$

Here, $\overline{A}_n$ is the average size of the left subtree and $\overline{B}_n$ the average size of the left subtree; it's clear that $\overline{A}_n + \overline{B}_n + 1 = n$.

The grammar translate into the equation system

$$\begin{aligned} S(\mathbf{p}) &= zA(\mathbf{p})B(\mathbf{p}) \\ A(\mathbf{p}) &= cazA(\mathbf{p})^2 + caz \\ B(\mathbf{p}) &= dbzB(\mathbf{p}) + dbz \end{aligned}$$

with $\mathbf{p} = (a, b, z)$, which solves to

$$S(a, b, z) = \frac{bdz - bdz \cdot \sqrt{1 - 4a^2c^2z^2}}{2ac - 2abcdz} .$$

Now, we obtain the ordinary generating function of $|L_n|$ by setting a and b to one, i. e.

$$L(z) = S[a = 1, b = 1](z) = \frac{dz}{2c(1 - dz)} - \frac{dz \cdot \sqrt{1 - 4c^2z^2}}{2c(1 - dz)} .$$

Those for A_n and B_n, i. e. the total number of nodes of type A resp. B in L_n, turn out to be

$$A(z) = \left(\tfrac{\partial}{\partial a} S[b = 1]\right)[a = 1](z) = -\frac{dz}{2c(1 - dz)} + \frac{dz}{2c(1 - dz) \cdot \sqrt{1 - 4c^2z^2}} \quad \text{and}$$

$$B(z) = \left(\tfrac{\partial}{\partial b} S[a = 1]\right)[b = 1](z) = \frac{dz}{2c(1 - dz)^2} - \frac{dz \cdot \sqrt{1 - 4c^2z^2}}{2c(1 - dz)^2} ,$$

respectively. All three functions have singularities $z_0 = 1/d$ and $z_{1,2} = \pm 1/2c$. As a consequence, we have to investigate three cases:

1. $d > 2c$ — lists dominate;

2. $d = 2c$ — equilibrium;

3. $d < 2c$ — trees dominate.

The cases differ in which of the singularities are dominant.

2.1 List Domination

Here we assume that $d > 2c$, i. e. $z_0 = {}^1\!/_d$ is the dominant singularity.

Towards $|L_n|$: We write

$$L(z) = \frac{dz}{2c(1-dz)} \cdot \left(1 - \sqrt{1 - 4c^2z^2}\right) .$$

Since the square root is defined in z_0 we can evaluate it at that point and read off the coefficients with standard identities; we get

$$|L_n| \sim d^n \cdot \frac{1 - \sqrt{1 - ({}^{2c}\!/_d)^2}}{2c} > 0 .$$

Towards $\overline{A}_n$: We write

$$A(z) = \frac{dz}{2c(1-dz)} \cdot \left(\frac{1}{\sqrt{1 - 4c^2z^2}} - 1\right) .$$

Since the square root is defined in z_0 we can evaluate it at that point and read off the coefficients with standard identities; we get

$$A_n \sim d^n \cdot \frac{\frac{1}{\sqrt{1-({}^{2c}\!/_d)^2}} - 1}{2c} > 0$$

and hence

$$\overline{A}_n \sim \frac{A_n}{|L_n|} = \frac{1}{\sqrt{1 - ({}^{2c}\!/_d)^2}} \in \Theta(1) .$$

Towards $\overline{B}_n$: We write

$$B(z) = \frac{dz}{2c(1-dz)^2} \cdot \left(1 - \sqrt{1 - 4c^2z^2}\right) .$$

Since the square root is defined in z_0 we can evaluate it at that point and read off the coefficients with standard identities; we get

$$B_n \sim d^n \cdot n \cdot \frac{1 - \sqrt{1 - ({}^{2c}\!/_d)^2}}{2c}$$

and hence

$$\overline{B}_n \sim \frac{B_n}{|L_n|} = n .$$

2.2 Equilibrium

Here we assume that $d = 2c$, i. e. all singularities are dominant. Note that we substitute $2c \mapsto d$ so as to simplify the generating functions; singularities are now $z_0 = z_1 = {}^1\!/_d$ and $z_2 = -{}^1\!/_d$.

Towards $|L_n|$: We write $L(z)$ as

$$L(z) = \frac{z}{1-dz} - \begin{cases} z \cdot \sqrt{1+dz} \cdot (1 - \frac{z}{z_1})^{-1/2} \\ \frac{z}{\sqrt{1-dz}} \cdot (1 - \frac{z}{z_2})^{1/2} \end{cases}$$

and thus conclude via DARBOUX's theorem [Kem85] that

$$\begin{aligned} |L_n| &\sim d^{n-1} - \frac{1}{n} \cdot \left[\frac{\frac{\sqrt{2}}{d} \cdot n^{1/2}}{\Gamma(1/2) \cdot z_1^n} + \frac{-\frac{1}{d\sqrt{2}} \cdot n^{-1/2}}{\Gamma(-1/2) \cdot z_2^n} \right] \\ &\sim d^{n-1} \, . \end{aligned}$$

Note that the first summand of $L(z)$ is not analytic in ${}^1\!/_d$ so we cannot "hide" it in (all) f_λ. We therefore treat the summands separately, the first with known identities and the second with DARBOUX's theorem [Kem85].

Towards $\overline{A}_n$: We write $A(z)$ as

$$A(z) = -\frac{z}{1-dz} + \begin{cases} \frac{z}{\sqrt{1+dz}} \cdot (1 - \frac{z}{z_1})^{-3/2} \\ \frac{z}{(1-dz)^{3/2}} \cdot (1 - \frac{z}{z_2})^{-1/2} \end{cases}$$

and thus conclude via DARBOUX's theorem [Kem85] that

$$\begin{aligned} A_n &\sim -d^{n-1} + \frac{1}{n} \cdot \left[\frac{\frac{1}{d\sqrt{2}} \cdot n^{3/2}}{\Gamma(3/2) \cdot z_1^n} + \frac{-\frac{1}{2d\sqrt{2}} \cdot n^{1/2}}{\Gamma(1/2) \cdot z_2^n} \right] \\ &\sim d^{n-1} \cdot \sqrt{n} \cdot \sqrt{\frac{2}{\pi}} \, . \end{aligned}$$

Hence we get that

$$\overline{A}_n = \frac{A_n}{|L_n|} \sim \sqrt{n} \cdot \sqrt{\frac{2}{\pi}} \, .$$

Towards $\overline{B}_n$: We write $B(z)$ as

$$B(z) = \frac{z}{(1-dz)^2} - \begin{cases} z \cdot \sqrt{1+dz} \cdot (1 - \frac{z}{z_1})^{-3/2} \\ \frac{z}{(1-dz)^{3/2}} \cdot (1 - \frac{z}{z_2})^{1/2} \end{cases}$$

and thus conclude via DARBOUX's theorem [Kem85] that

$$B_n \sim d^{n-1} \cdot n - \frac{1}{n} \cdot \left[\frac{\frac{\sqrt{2}}{d} \cdot n^{3/2}}{\Gamma(3/2) \cdot z_1^n} + \frac{-\frac{1}{2d\sqrt{2}} \cdot n^{-1/2}}{\Gamma(-1/2) \cdot z_2^n} \right]$$
$$\sim d^{n-1} \cdot n \, .$$

Hence we get that

$$\overline{B}_n = \frac{B_n}{|L_n|} \sim n \, .$$

So it's not an equilibrium, after all.

2.3 Tree Domination

Here we assume that $d < 2c$, i. e. $z_{1,2} = \pm 1/2c$ are the dominant singularities.

Towards $|L_n|$: We write $L(z)$ as

$$L(z) = \frac{dz}{2c(1-dz)} - \begin{cases} \frac{dz \cdot \sqrt{1+2cz}}{2c(1-dz)} \cdot (1 - \frac{z}{z_1})^{1/2} \\ \frac{dz \cdot \sqrt{1-2cz}}{2c(1-dz)} \cdot (1 - \frac{z}{z_2})^{1/2} \end{cases}$$

and thus conclude via DARBOUX's theorem [Kem85] that

$$|L_n| \sim \frac{1}{n} \cdot \left[\frac{-\frac{d}{c\sqrt{2}(2c-d)} \cdot n^{-1/2}}{\Gamma(-1/2) \cdot z_1^n} + \frac{\frac{d}{c\sqrt{2}(2c+d)} \cdot n^{-1/2}}{\Gamma(-1/2) \cdot z_2^n} \right]$$
$$= \frac{(2c)^n}{n^{3/2}} \cdot \left[\frac{d}{2c\sqrt{2\pi}(2c-d)} - \frac{d}{2c\sqrt{2\pi}(2c+d)} \cdot (-1)^n \right]$$
$$= \frac{(2c)^n}{n^{3/2}} \cdot \begin{cases} \frac{d^2}{(4c^2-d^2)c\sqrt{2\pi}}, & n \text{ even} \\ \frac{d\sqrt{2}}{(4c^2-d^2)\sqrt{\pi}}, & n \text{ odd} \end{cases} .$$

Towards $\overline{A}_n$: We write $A(z)$ as

$$A(z) = -\frac{dz}{2c(1-dz)} + \begin{cases} \frac{dz}{2c(1-dz) \cdot \sqrt{1+2cz}} \cdot (1 - \frac{z}{z_1})^{-1/2} \\ \frac{dz}{2c(1-dz) \cdot \sqrt{1-2cz}} \cdot (1 - \frac{z}{z_2})^{-1/2} \end{cases}$$

and thus conclude via DARBOUX's theorem [Kem85] that

$$
\begin{aligned}
A_n &\sim \frac{1}{n} \cdot \left[\frac{\frac{d}{2c\sqrt{2}(2c-d)} \cdot n^{1/2}}{\Gamma(1/2) \cdot z_1^n} + \frac{-\frac{d}{2c\sqrt{2}(2c+d)} \cdot n^{1/2}}{\Gamma(1/2) \cdot z_2^n} \right] \\
&= \frac{(2c)^n}{\sqrt{n}} \cdot \left[\frac{d}{2c\sqrt{2\pi}(2c-d)} - \frac{d}{2c\sqrt{2\pi}(2c+d)} \cdot (-1)^n \right] \\
&= \frac{(2c)^n}{\sqrt{n}} \cdot \begin{cases} \frac{d^2}{(4c^2-d^2)c\sqrt{2\pi}}, & n \text{ even} \\ \frac{d\sqrt{2}}{(4c^2-d^2)\sqrt{\pi}}, & n \text{ odd} \end{cases} .
\end{aligned}
$$

Hence we get that

$$
\overline{A}_n = \frac{A_n}{|L_n|} \sim n .
$$

Towards $\overline{B}_n$: We write $B(z)$ as

$$
B(z) = \frac{dz}{2c(1-dz)^2} - \begin{cases} \frac{dz \cdot \sqrt{1+2cz}}{2c(1-dz)^2} \cdot (1 - \frac{z}{z_1})^{1/2} \\ \frac{dz \cdot \sqrt{1-2cz}}{2c(1-dz)^2} \cdot (1 - \frac{z}{z_2})^{1/2} \end{cases}
$$

and thus conclude via DARBOUX's theorem [Kem85] that

$$
\begin{aligned}
B_n &\sim \frac{1}{n} \cdot \left[\frac{-\frac{d\sqrt{2}}{(d-2c)^2} \cdot n^{-1/2}}{\Gamma(-1/2) \cdot z_1^n} + \frac{\frac{d\sqrt{2}}{(d+2c)^2} \cdot n^{-1/2}}{\Gamma(-1/2) \cdot z_2^n} \right] \\
&= \frac{(2c)^n}{n^{3/2}} \cdot \left[\frac{d}{\sqrt{2\pi}(d-2c)^2} - \frac{d}{\sqrt{2\pi}(d+2c)^2} \cdot (-1)^n \right] \\
&= \frac{(2c)^n}{n^{3/2}} \cdot \begin{cases} \frac{4\sqrt{2}cd^2}{(4c^2-d^2)^2\sqrt{\pi}}, & n \text{ even} \\ \frac{d\sqrt{2}(4c^2+d^2)}{(4c^2-d^2)\sqrt{\pi}}, & n \text{ odd} \end{cases} .
\end{aligned}
$$

Hence we get that

$$
\overline{B}_n = \frac{B_n}{|L_n|} \sim \begin{cases} \frac{8c^2}{4c^2-d^2}, & n \text{ even} \\ \frac{4c^2+d^2}{4c^2-d^2}, & n \text{ odd} \end{cases} \in \Theta(1) .
$$

In the domination picture, all is as expected; one side takes up about all the nodes and hence its average height carries over. In the equilibrium case, however, we notice that $\overline{A}_n \in \Theta(\sqrt{n})$. Are there more possible behaviours?

3 Example Md

We follow a suggestion by Michael Drmota towards the following example. It derives directly from Example MBM in the Equilibrium case because we want to reproduce the switch of dominance between pole and algebraic singularity we observed there (when taking the derivative w. r. t. a). Hence, we define

$$S \to \begin{array}{c} \mathrm{s} \\ / \ \backslash \\ S' \ \ B \end{array}$$

$$B \to \begin{array}{c} a_1 \\ | \\ \mathrm{B} \end{array} \Bigg| \cdots \Bigg| \begin{array}{c} a_d \\ | \\ \mathrm{B} \end{array} \Bigg| \, a_1 \Bigg| \cdots \Bigg| \, a_d$$

where S' would be the starting symbol of a grammar for a language L' with asymptotic number of trees

$$|L'_n| \sim c \cdot d^n \cdot n^\alpha$$

for some $d > 0$, $c > 0$ and $\alpha \in \mathbb{D}_2$ where

$$\mathbb{D}_2 = \left\{ -1 - \frac{1}{2^k} \;\middle|\; k \geq 1 \right\} \cup \left\{ \frac{m}{2^k} - 1 \;\middle|\; m \geq 1, k \geq 0 \right\} .$$

Such a grammar exists for all α (and suitable c, d) according to Banderier and Drmota [BD14]. Note that the equation systems given in the proof of Proposition 6 of that article readily translate into regular tree grammars. However, the resulting generating functions are only available in implicit form, i. e. all but intractable for our purposes.

Therefore we employ Corollary 1 which promises that as long as we use *any* generating function with the same coefficient-asymptotics as the one from Banderier and Drmota [BD14][2] we get a Θ-asymptotic for $|L_n|$. The same argument applies to the partial derivative, in particular because of Lemma 2.

Hence, we perform the calculations just like above with

$$z \cdot \left(1 \pm (1 - daz)^{-\alpha-1}\right) \cdot \frac{z}{1 - dz}$$

or rather – since we can only obtain Θ-bounds, anyway – with

$$S(a, z) = \left(1 \pm (1 - daz)^{-\alpha-1}\right) \cdot \frac{1}{1 - dz}$$

for the sake of simplicity. Note that $\pm$ has to be chosen depending on α so that all coefficients are positive; we need $+$ if $\Gamma(\alpha + 1) < 0$ and $-$ otherwise.

[2] There may be finitely many $f_i = 0$ in the "language" from Banderier and Drmota [BD14]; these we can fix by "hardcoding" corresponding trees without hurting the asymptotics. Hence we find a language whose generating function fulfills the conditions for Lemma 1.

We obtain, following the terminology from above,

$$L(z) = \frac{1}{1-dz} \pm \frac{1}{(1-dz)^{\alpha+2}} \quad \text{and}$$

$$A(z) = \pm(1+\alpha) \cdot d \cdot \frac{z}{(1-dz)^{\alpha+3}}$$

Straight-forward application of Darboux's theorem [Kem85] yields

$$|L_n| \in \begin{cases} \Theta(d^n) & ,\alpha < -1 \\ \Theta(d^n \cdot n^{\alpha+1}) & ,\alpha > -1 \end{cases} \quad \text{and}$$

$$A_n \in \Theta(d^n \cdot n^{\alpha+2}) \, .$$

Recall that $-1 \notin \mathbb{D}_2$ so the case distinction is complete. Finally, we obtain that

$$\overline{A}_n \in \begin{cases} \Theta(n^{\alpha+2}) & ,\alpha < -1 \\ \Theta(n) & ,\alpha > -1 \end{cases} \, .$$

We conclude that for each

$$\beta \in \left\{ 1 - \frac{1}{2^k} \;\middle|\; k \geq 1 \right\}$$

there is a regular tree grammar with a non-terminal A for which

$$\overline{A}_n \in \Theta(n^\beta) \, .$$

Hence our investigation on this front has come to an end.

4 Auxiliary Results

Theorem 1 (DARBOUX's theorem [Kem85])**.** *Let* $A(z) = \sum_{n\geq 0} a_n z^n$ *be a function with radius of convergence* $\rho_A > 0$ *and a finite number of singularities* $z_\lambda, \lambda \in [1:m]$*, on the circle of convergence* $|z| = \rho_A$*. If there is an expansion of the form*

$$A(z) = f_\lambda(z) + g_\lambda(z) \cdot \left(1 - \frac{z}{z_\lambda}\right)^{-\omega_\lambda}, \quad \lambda \in [1:m] ,$$

around every singularity z_λ *where*

1. f_λ *and* g_λ *are analytic near* z_λ*,*
2. g_λ *is non-zero near* z_λ *and*
3. $-\omega_\lambda \in \mathbb{C} \setminus \mathbb{N}_0$*,*

then we have

$$a_n = \frac{1}{n} \cdot \sum_{\lambda=1}^{m} \frac{g_\lambda(z_\lambda) \cdot n^{\omega_\lambda}}{\Gamma(\omega_\lambda) \cdot z_\lambda^n} + o(\rho_A^{-n} \cdot n^{w-1})$$

with $w = \max_{1\leq\lambda\leq m} \Re(\omega_\lambda)$*.*

Lemma 1 (Θ-stability of (some) convolutions)**.** *Let*

- $g(z) = f(z) \cdot (1 - dz)^{-1}$ *with*
- $n \mapsto f_n \in \mathbb{N}_+ \to \mathbb{N}_+$ *and,*
- $f_n \sim c \cdot d^n \cdot n^\alpha$ *where*
- $\alpha \in \mathbb{D}_2$ *[BD14],* $c > 0$ *and* $d \geq 1$*.*

Restriction on α unnecessary?

Then,

$$a_n := \sum_{i=1}^{n} c \cdot d^i \cdot i^\alpha \cdot d^{n-i} \in \Theta(g_n) .$$

Proof. Let w. l. o. g. $c = 1$; the constant factor can be pulled out of the sum and does not change the calculations.

First, we unwrap the definition of $f_n \sim \dots$ into

$$\forall \epsilon > 0 \, \exists n_0 > 0 \, \forall n \geq n_0 \, . \quad \frac{f_n}{1+\epsilon} \leq d^n n^\alpha \tag{1}$$

$$\wedge \Big[\epsilon < 1 \implies d^n n^\alpha \leq \frac{f_n}{1-\epsilon} \Big] \, . \tag{2}$$

Now we show for two constants $c_l, c_u > 0$ that

i) $a_n \leq c_u \cdot g_n$ and

ii) $a_n \geq c_l \cdot g_n$;

the claim then follows immediately.

ad i): Let $\epsilon \in (0,1)$ arbitrary and n_0 accordingly. We calculate:

$$\begin{aligned}
a_n &= \sum_{i=1}^{n} d^i \cdot i^\alpha \cdot d^{n-i} \\
&= \sum_{i=1}^{n_0} d^i \cdot i^\alpha \cdot d^{n-i} + \sum_{i=n_0+1}^{n} d^i \cdot i^\alpha \cdot d^{n-i} \\
&\overset{(2)}{\leq} d^n \cdot \underbrace{\begin{cases} n_0^{\alpha+1} & , \alpha \geq 0 \\ n_0 & , \alpha < 0 \end{cases}}_{=:b} + \sum_{i=n_0+1}^{n} \frac{f_i}{1-\epsilon} \cdot d^{n-i} \\
&\leq b \cdot d^n + \frac{1}{1-\epsilon} \cdot \sum_{i=1}^{n} f_i \cdot d^{n-i} \\
&\overset{f_i > 0}{\leq} \frac{db}{f_1} \cdot \sum_{i=1}^{n} f_i \cdot d^{n-i} + \frac{1}{1-\epsilon} \cdot \sum_{i=1}^{n} f_i \cdot d^{n-i} \\
&= \underbrace{\left(\frac{db}{f_1} + \frac{1}{1-\epsilon} \right)}_{=:c_u(\epsilon)} \cdot g_n \, .
\end{aligned}$$

The claim of i) follows with e. g. $c_u = c_u(1/2) = 2 + dbf_1^{-1}$ which is indeed a constant for fixed f and ϵ.

ad ii): For the time being, let $\epsilon > 0$ arbitrary and calculate:

$$\begin{aligned}
a_n &= \sum_{i=1}^{n} d^i \cdot i^\alpha \cdot d^{n-i} \\
&= \sum_{i=1}^{n_0} d^i \cdot i^\alpha \cdot d^{n-i} + \sum_{i=n_0+1}^{n} d^i \cdot i^\alpha \cdot d^{n-i} \\
&\overset{(1)}{\geq} \sum_{i=1}^{n_0} d^i \cdot i^\alpha \cdot d^{n-i} + \sum_{i=n_0+1}^{n} \frac{f_i}{1+\epsilon} \cdot d^{n-i} \\
&= \sum_{i=1}^{n_0} d^i \cdot i^\alpha \cdot d^{n-i} + \frac{1}{1+\epsilon} \cdot g_n - \sum_{i=1}^{n_0} \frac{f_i}{1+\epsilon} \cdot d^{n-i} \\
&= \sum_{i=1}^{n_0} d^{n-i} \cdot \underbrace{\left[d^i \cdot i^\alpha - \frac{f_i}{1+\epsilon}\right]}_{\geq 0 \ (\star)} + \frac{1}{1+\epsilon} \cdot g_n \\
&\geq \underbrace{\frac{1}{1+\epsilon}}_{=:c_l} \cdot g_n
\end{aligned}$$

and the claim of ii) is shown.

Now, $(\star)$ does certainly not hold for *all* $\epsilon > 0$ (and f) so more care has to be taken in picking a suitable one. Let us start with an arbitrary $\epsilon' > 0$ and its corresponding n_0'. Now the inequality in question,

$$d^i \cdot i^\alpha - \frac{f_i}{1+\epsilon'} \geq 0 \,,$$

holds for all $n \geq n_0'$ because of (1). Since $f_i > 0$ for all i, this does not change if we increase ϵ'; the second, negative summand just becomes even smaller. Intuitively speaking, "increasing ϵ' moves n_0' towards 0", and that is exactly what we need. Thus we increase ϵ' to

$$\epsilon := \frac{\max_{1 \leq i \leq n_0'} f_i}{\min\{d^i i^\alpha \mid 1 \leq i \leq n_0'\} \cup \{1\}} - 1 \,,$$

if it is not already larger; in that case, we stick with $\epsilon = \epsilon'$. Note that $\epsilon > 0$ assuming that $\max_{1 \leq i \leq n_0'} f_i > 1$. This we can always achieve by increasing n_0' in the first place; due to $f_n \sim d^n n^\alpha$ we are sure to find $f_i > 1$ eventually. For completeness, note that we can now pick n_0 arbitrarily, in particular $n_0 = n_0'$.

Or by $\epsilon \geq \epsilon' > 0$.

For good measure, we quickly check that, indeed, for all $i \in [1..n_0]$

$$
\begin{aligned}
d^i \cdot i^\alpha - \frac{f_i}{1+\epsilon} &\geq d^i \cdot i^\alpha - \frac{f_i}{\frac{\max_{1 \leq i \leq n_0'} f_i}{\min\{d^i i^\alpha | 1 \leq i \leq n_0'\} \cup \{1\}}} \\
&\geq d^i \cdot i^\alpha - \min\{d^i i^\alpha \mid 1 \leq i \leq n_0\} \cup \{1\} \\
&\geq 0 \ .
\end{aligned}
$$

Hence, we can indeed perform above calculation with this here ϵ. Note that, despite all the fiddling, ϵ is still a constant for fixed f.

Is above calculation even necessary with this ϵ? Splitting the sum is no longer necessary since $n_0 = 1$ is possible, isn't it?

Having shown both i) and ii) we have completed the proof of Lemma 1. □

Corollary 1. *Let g and f as in Lemma 1 and additionally $g'(z) = f'(z) \cdot (1 - dz)^{-1}$ with $f_n' \sim f_n$. Then,*

$$g_n \in \Theta(g_n') \ .$$

Proof. Since

$$f_n \sim c \cdot d^n \cdot n^\alpha \sim f_n' \ ,$$

we can apply Lemma 1 on both g and g' and the claim follows with transitivity of Θ-equality. □

Since we will use above results not only on "original" generating functions but also on their derivatives, let us note the following.

Lemma 2 (Asymptotics survive derivation)**.** *Let f and g be generating functions with $f_n \sim g_n$. Then,*

$$[z^n]\Big(\tfrac{\mathrm{d}}{\mathrm{d}z}\, f(z)\Big) \sim [z^n]\Big(\tfrac{\mathrm{d}}{\mathrm{d}z}\, g(z)\Big) \ .$$

Proof.

$$[z^n]\Big(\tfrac{\mathrm{d}}{\mathrm{d}z}\, f(z)\Big) = (n+1) \cdot f_{n+1} \sim (n+1) \cdot g_{n+1} = [z^n]\Big(\tfrac{\mathrm{d}}{\mathrm{d}z}\, g(z)\Big) \ .$$

□

Lemma 3. *Let $L \in \mathsf{TREG}$. Then there is an* unambiguous *regular tree grammar G with $L(G) = L$.*

Sketch of Proof. Assume towards a contradiction that there was $L \in \mathsf{TREG}$ so that every regular grammar for L was ambiguous.

We know from Comon et al. [Com+07, Theorems 1.1.9, 1.6.1 and 2.1.5] that L is accepted by a *deterministic* bottom-up regular tree automaton A. Following the constructions given (or outlined) in the proofs of the corresponding theorems, we obtain via a (non-deterministic) top-down regular tree automaton a regular tree grammar G with $L(G) = L$. By assumption G is ambiguous, that is there are at least two distinct left-derivations in G for some tree $t \in L$.

It is easy to see from the syntactic nature of the constructions that these derivations induce distinct accepting computations for t in A which contradicts that A is deterministic; hence the assumption must be false. ⬚

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